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A mean value theorem on Dirichlet series

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Abstract. This paper deals with mean-value for the square of certain function $F(s)$ which has some characteristic properties of the Riemann zeta-function and its powers.

Keywords: Dirichlet series, mean value, Riemann hypothesis.

1. Introduction

In [1] Balasubramanian, Ivić and Ramachandra obtained mean-value theorems for certain functions $F(s) = \sum a_n \lambda_n^{-s}$ over intervals of the line $Res = 1$. Sankaranarayanan and Srinivas in [5] proved a mean-value theorem of the Riemann zeta-function over shorter intervals on the assumption of the Riemann Hypothesis.

In this paper we obtain a mean-value theorem for functions $F(s)$ which verify the following conditions:

i) $F(s)$ is a meromorphic function in the half-plane $Res > 1/2$ with at most one pole in order r at $s = 1$ with residue γ; $F(s)$ has a representation as a Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s} \tag{1.1}$$

absolutely convergent in $Res > 1$ and $f(n) \ll n^{\epsilon}$ for every $\epsilon > 0$. For every $\sigma' > 1/2$

$$F(\sigma + it) = O(|t|^{C}) \qquad (\text{as } |t| \to \infty) \tag{1.2}$$

uniformly in $\sigma \geq \sigma'$, with an appropriate constant C depending on σ'.

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ii) Furthermore, we suppose that the function defined as

$$G(s) = \sum_{n=1}^{\infty} |f(n)|^2 n^{-s} \tag{1.3}$$

in $Res > 1$ has an analytic continuation over certain half-plane $Res > a$ $(0 < a < 1)$ and its only singularity is a pole of order m at $s = 1$, with residue c_m.

iii) Besides, there exist positive constants A_1, β such that for

$$\frac{1}{2} + \frac{A_1}{(\log\log t)^{\beta}} \leq \sigma \leq 1 - \delta \qquad (\delta > 0), \qquad t \geq t_0 > 0 \tag{1.4}$$

we have uniformly

$$\log F(\sigma + it) = O((\log t)^{\alpha(\sigma)} (\log\log t)^{-\beta}) \tag{1.5}$$

where $0 < \alpha(\sigma) < 1$ and $\alpha(\sigma)$ is a continuous and decreasing function for $1/2 < \sigma < 1$.

It is the object of this paper to prove a mean-value theorem for the square of $F(s)$ over short intervals; so, we will prove the following result:

Theorem. *For*

$$\frac{1}{2} + \frac{2A_1}{(\log\log t)^{\beta}} \leq \sigma \leq 1 - \delta \qquad (\delta > 0) \tag{1.6}$$

and $Y \leq H \leq T$ *with* $Y = \exp[(\log T)^{\alpha(\sigma)}]$ *we have*

$$\begin{aligned} \frac{1}{H}\int_T^{T+H} |F(s)|^2 dt = \sum_{n=1}^{\infty} |f(n)|^2 n^{-2\sigma} + \\ + O\left(\exp(-A_2 (\log T)^{\alpha(\sigma)} (\log\log T)^{-\beta})\right) \end{aligned} \tag{1.7}$$

where $A_2 > 0$ *depends on* A_1 *and the* O*-constant depends on* δ.

2. Lemmas

Lemma 1. *Under the hypothesis of the Theorem and* $T \leq t \leq T + H$ *we have*

$$F(s) = \sum_{n=1}^{\infty} f(n) e^{-n/Y} n^{-s} + O\left(\exp(-A_3 (\log T)^{\alpha(\sigma)} (\log\log T)^{-\beta})\right) \tag{2.1}$$

where $A_3 > O$ *depends on* A_1.

Proof. By Mellin's transformation,

$$\sum_{n=1}^{\infty} f(n)e^{-n/Y}n^{-s} = \frac{1}{2\pi i}\int_{Re\omega=2} F(s+\omega)Y^{\omega}\Gamma(\omega)d\omega$$
$$= \frac{1}{2\pi i}\int_{\substack{Re\omega=2\\ |v|>\log^a T}} F(s+\omega)Y^{\omega}T(\omega)d\omega + \tag{2.2}$$
$$+ \frac{1}{2\pi i}\int_{\substack{Re\omega=2\\ |v|\leq\log^a T}} F(s+\omega)Y^{\omega}T(\omega)d\omega$$

where $a > 0$ is sufficiently large and $\omega = u + iv$.

We consider the rectangle R_T with vertices

$$2 \pm i\log^a T, (-A_1(\log\log T)^{-\beta}) \pm i\log^a T,$$

and by the Cauchy residue theorem we can write:

$$\sum_{n=1}^{\infty} f(n)e^{-n/Y}n^{-s} = \sum_{R_T} Res(F(s+\omega)Y^{\omega}\Gamma(\omega))+$$
$$+O\left(\left|\int_{\substack{Re\omega=2\\ |v|\geq\log^a T}} F(s+\omega)Y^{\omega}\Gamma(\omega)d\omega\right|\right)+$$
$$+O\left(\left|\int_{-A_1(\log\log T)^{-\beta})+i\log^a T}^{2+i\log^a T} F(s+\omega)Y^{\omega}\Gamma(\omega)d\omega\right|\right)+ \tag{2.3}$$
$$+O\left(\left|\int_{\substack{Re\omega=-A_1(\log\log T)^{-\beta}\\ |v|\leq\log^a T}} F(s+\omega)Y^{\omega}\Gamma(\omega)d\omega\right|\right).$$

By the well-known estimation of $\Gamma(\omega)$ (see A.3 [**2**]) we obtain

$$\int_{\substack{Re\omega=2\\ |v|\geq\log^a T}} F(s+\omega)Y^{\omega}\Gamma(\omega)d\omega = O\left(Y^2\int_{\substack{Re\omega=2\\ |v|\geq\log^a T}} |\Gamma(\omega)|d\omega\right)$$
$$= O\left(Y^2\int_{v>\log^a T} v^{3/2}e^{-\pi v/2}dv\right) \tag{2.4}$$
$$= O(Y^2\exp(-C_1\log^a T))$$

with $C_1 > 0$.

From the condition (1.2) and the estimation for $\Gamma(\omega)$ we can write:

$$\begin{aligned}\int_{-A_1(\log\log T)^{-\beta}+i\log^a T}^{2+i\log^a T} F(s+\omega)Y^\omega\Gamma(\omega)d\omega &= O(T^c Y^2\exp(-C_2\log^a T)) \\ &= O(Y^2\exp(-C_3\log^a T))\end{aligned} \tag{2.5}$$

with C_2, $C_3 > 0$.

By the hypothesis iii) for $F(s)$ we have

$$\int_{\substack{Re\omega=-A_1(\log\log T)^{-\beta}\\ |v|\le\log^a T}} F(s+\omega)Y^\omega\Gamma(\omega)d\omega =$$

$$= O\left(\frac{\exp\left\{C_4\frac{(\log T)^{\alpha(\sigma)}}{(\log\log T)^\beta}\right\}}{Y^{A_1(\log\log T)^{-\beta}}}\int_{\substack{Re\omega=-A_1(\log\log T)^{-\beta}\\ |v|\le\log^a T}}\left|\frac{\Gamma(1+\omega)}{\Gamma(\omega)}\right|d\omega\right) = \tag{2.6}$$

$$= O(Y^{-A_1(\log\log T)^{-\beta}}\exp(C_4(\log T)^{\alpha(\sigma)}(\log\log T)^{-\beta})\log\log T).$$

Choosing $Y=\exp((\log T)^{\alpha(\sigma)})$ the three integrals of (2.3) are

$$O(\exp\{-A_3(\log T)^{\alpha(\sigma)}(\log\log T)^{-\beta}\}) \tag{2.7}$$

for some $A_3 > 0$ sufficiently large.

Besides, $\Gamma(s)$ has a single pole at $\omega = 0$ with residue 1; so

$$Res_{\omega=0}(F(s+\omega)Y^\omega\Gamma(\omega)) = F(s). \tag{2.8}$$

At $\omega = 1-s$, $F(s+\omega)$ has a pole of order r and by the estimation of $\Gamma(\omega)$, the residue of $F(s+\omega)Y^\omega\Gamma(\omega)$ in this point is bounded by (2.7).

Lemma 2. [2,4] *If* $\{a_n\}$ *is a sequence of complex numbers such that* $\sum n(|a_n|)^2$ *is convergent, then*

$$\int_T^{T+H}\left|\sum_{n=1}^\infty a_n n^{-it}\right|^2 dt = \sum_{n=1}^\infty |a_n|^2(H+O(n)) \tag{2.9}$$

where the implied constant is absolute.

3. Proof of Theorem

Let $Y = \exp((\log T)^{\alpha(\sigma)})$ and $T \leq t \leq T + H$. By Lemma 1 we have

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} f(n) e^{-n/Y} n^{-s} + O(\exp\{-A_3 (\log T)^{\alpha(\sigma)} (\log\log T)^{-\beta}\}) \\ &= J_1 + J_2 \end{aligned} \tag{3.1}$$

when

$$\frac{1}{2} + \frac{2A_1}{(\log\log T)^{\beta}} \leq \sigma \leq 1 - \delta.$$

So,

$$\int_T^{T+H} |F(s)|^2 dt = \int_T^{T+H} (|J_1|^2 + |J_2|^2 + J_1 \bar{J}_2 + \bar{J}_1 J_2) dt. \tag{3.2}$$

By Lemma 2,

$$\begin{aligned} \int_T^{T+H} |J_1|^2 dt &= \sum_{n=1}^{\infty} |f(n)| e^{-2n/Y} n^{-2\sigma} (H + O(n)) \\ &= H \left\{ \sum_{n=1}^{\infty} |f(n)|^2 n^{-2\sigma} + \sum_{n \leq Y} |f(n)|^2 n^{-2\sigma} (e^{-2n/Y} - 1) + \right. \\ &\quad \left. + \sum_{n \geq Y} |f(n)|^2 n^{-2\sigma} (e^{-2n/Y} - 1) \right\} + \\ &\quad + O\left(\sum_{n=1}^{\infty} |f(n)|^2 n^{1-2\sigma} e^{-2n/Y} \right). \end{aligned} \tag{3.3}$$

Now,

$$\sum_{n \leq Y} |f(n)|^2 n^{-2\sigma} (e^{-2n/Y} - 1) = O\left(Y^{-1} \sum_{n \leq Y} n^{1-2\sigma+\epsilon} \right) = O\left(\frac{Y^{1-2\sigma+\epsilon}}{2 - 2\sigma + \epsilon} \right)$$

$$\sum_{n > Y} |f(n)|^2 n^{-2\sigma} = O\left(\frac{Y^{1-2\sigma+\epsilon}}{2 - 2\sigma + \epsilon} \right)$$

$$\sum_{n > Y} |f(n)|^2 n^{-2\sigma} e^{-2n/Y} = O\left(Y \sum_{n > Y} n^{-1-2\sigma+\epsilon} \right) = O\left(\frac{Y^{1-2\sigma-\epsilon}}{2\sigma - \epsilon} \right)$$

$$\sum_{n=1}^{\infty} |f(n)|^2 n^{1-2\sigma} e^{-2n/Y} = O\left(\frac{Y^{2-2\sigma+\epsilon}}{2\sigma - 1 + \epsilon}\right).$$

Since $Y \leq H$ we get

$$\int_T^{T+H} |J_1|^2 dt = = H\left\{\sum_{n=1}^{\infty} |f(n)|^2 n^{-2\sigma} + O\left(\frac{Y^{1-2\sigma+\epsilon}}{2-2\sigma+\epsilon}\right) + O\left(\frac{Y^{1-2\sigma+\epsilon}}{2\sigma}\right)\right\}. \tag{3.4}$$

From (3.1), obviously

$$\int_T^{T+H} |J_2|^2 dt = O(H \exp\{-2A_3 (\log T)^{\alpha(\sigma)} (\log\log T)^{-\beta}\}). \tag{3.5}$$

Besides, by using Holder's inequality

$$\int_T^{T+H} J_1 \bar{J}_2 dt = = O\left(H \left(\sum_{n=1}^{\infty} \frac{|f(n)|^2}{n^{2\sigma}}\right)^{1/2} \exp\{-A_3 (\log T)^{\alpha(\sigma)} (\log\log T)^{-\beta}\}\right). \tag{3.6}$$

From (1.6) and ii) we deduce

$$\sum_{n=1}^{\infty} |f(n)|^2 n^{-2\sigma} \leq \sum_{n=1}^{\infty} |f(n)|^2 n^{-1-4A_1(\log\log T)^{-\beta}}$$
$$= \frac{c_m}{(4A_1(\log\log T)^{-\beta})^m} + O((4A_1(\log\log T)^{-\beta})^{1-m})$$

Therefore

$$\int_T^{T+H} J_1\bar{J}_2 dt = O(H(\log\log T)^{\frac{\beta m}{2}} \exp(-A_3(\log T)^{\alpha(\sigma)}(\log\log T)^{-\beta}))$$
$$= O(H\exp(-A_4(\log T)^{\alpha(\sigma)}(\log\log T)^{-\beta})), \tag{3.7}$$

where $A_4 < A_3$. Taking $Y = \exp((\log T)^{\alpha(\sigma)}$ Theorem is proved with $A_2 > 0$ such that $A_2 < A_3$ and $A_2 < 4A_1$. □

4. Applications

The functions

1) $\{F(s) = \zeta(s), \quad f(n) \equiv 1\}$, $\{F(s) = \zeta^k(s), \quad f(n) \equiv d_k(n), k \geq 2\}$,
2) $\{F(s) = L^k(s, \chi_0), \quad f(n) \equiv d_k(n)\chi_0(n)$, being χ_0 a principal character mod $q, q > 1, k \geq 1\}$.

3) $\{F(s) = \frac{\zeta^{k^m}(s)}{(\zeta(2s))^{c_0}} h_1(s), \quad f(n) \equiv (d_k(n))^m$, (where $h_1(s)$ is defined in the half-plane $Res > 1/3$ by an absolutely convergent Dirichlet series and $c_0 = (1/2)k^m(k^m+1) - ((k/2)(k+1))^m \in \mathbb{N}$ (see [**3**])}, hold the hypothesis i) and ii). The hypothesis iii) is verified by $F(s) = \zeta(s), F(s) = \zeta^k(s)$ and $F(s) = L^k(s, \chi_0)$ on the Riemann Hypothesis (see [**6**]).

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